Quantum categories. Quantization of the category of linear spaces.

Theodore Voronov

Moscow State University
Department of Mechanics and Mathematics
Chair of Higher Geometry and Topology
Vorobyovy Gory, Moscow 119899, RUSSIA
E-mail theodore@nw.math.msu.su, theodore@mech.math.msu.su

Abstract

We generalize the notion of bialgebras or Hopf algebras and on this basis we define quantum categories. The quantization of the category of linear (super)spaces is constructed, with the examples. We establish a criterion for the classical value of the dimension of the polynomial function algebra on the full quantum subcategory defined by the Sudbery type commutational relations for quantum vector spaces (the "Poincare – Birkhoff – Witt property"). The criterion is the equality of the "quantum constants" $c_\alpha = c_\beta^{\pm 1}$ for quantum spaces in consideration. We also establish links with the Yang – Baxter equation and the Yang – Baxter structures of quantum linear spaces. The role of categories as a generalization of groups and the related topic of a "category programme" are discussed in the introduction.

Introduction

Motivation.

The heuristic meaning of quantum group theory consists in the consideration of transformations that depend on non-commuting parameters. The parameters serve as generators of a certain (noncommutative) algebra, which is interpreted as the "function algebra" of the quantum group. The group nature of the transformations is formalized in the assumption that this algebra is endowed with the Hopf structure. In matrix case the Hopf structure is induced just by the standard matrix multiplication and by taking matrix inverse. But the transformations of the group nature (and the related algebraic structures) are not the only possible. Quantum semigroups appear along with quantum groups (at least as the intermediate product, see by Buchstaber and Rees [4]. For Novikov's operator doubles (which are not Hopf algebras) the "Hopf type" questions can be studied Odessky – Feigin algebras [25]. All this formally exceeds the boundaries of quantum groups but nevertheless is in coherence with the heuristic thesis stated above.

In the present paper we introduce and study QUANTUM CATEGORIES and the simplest but fundamental example, the QUANTIZATION OF THE CATEGORY OF LINEAR (SUPER)SPACES.

This needs to be somehow explained since it is very common to treat categories only as "general nonsense" (natural transformations of natural constructions [31]), a language necessary in algebraic topology and homological algebra but mostly for the purpose of systematization. The quantum group theorists also use categories in this manner. (Categories of representations and their abstract generalizations like "tensor" or "braided" categories are good example.) But the philosophy of categories in this paper is quite different. We consider categories first of all as algebraic structures, sets of arrows with the multiplication law. Hence categories are an immediate generalization of groups, for many reasons more useful than semigroups. Actually, this structure appears quite often in very classical situations. The category structure of such examples is of course known but usually is not stressed. (Paths on a topological space form a category. Parallel transports in fiber bundles and the homology "local coefficients" are examples of its representations. Cobordism provide examples of categories, with "films" as arrows. This includes "histories" or Lorentzian cobordism, see [20], [10]. It is worth mentioning that the coordinate changes in the fixed open domain $U \subset \mathbb{R}^{\kappa}$ also form a category, not a group. Its representations exactly correspond to "geometrical objects".) Recent research supplied new examples of categories naturally appearing in the context very far from "general nonsense". For example, it was found that the representations of classical groups such as spinor representation naturally extend to certain categories of linear relations [22], [23]. The study of Poisson geometry lead to symplectic groupoids [15], which provide an example of "continuous categories". The category of tangles [29] can be mentioned also. This list can be extended.

A conclusion is that a slightly different (from popular) view on general nonsense itself makes sense. The role of categories should be compared to that of groups in Klein's classification of geometries. (This view actually was expressed by the founders of category theory in [31].) Hence the main thesis of a *"category programme"* is as follows. *Categories are the most important generalization of groups; it is necessary to study their actions, representations, deformations and extensions, cohomology* (see [1], [30], [16]), *the analogues of Lie theory* etc. Actually the research in these directions is going on, with various motivations. We can especially mark [23] and [15]. (The author himself was brought to the necessity of a programme like this through the reflections on some problems of infinite-dimensional geometry connected with the topics of [7].) The study of quantum categories initiated in the present paper can be considered a step in the realization of the "category programme".

Inside the quantum group theory the particular motivations for introducing quantum categories are the following. The same classical object can admit different "quantum deformations" (for example, corresponding to different values of the deformation parameter). Which should be chosen ? The correct answer is that all of them should be considered simultaneously, together with all possible transformations between them. This inevitably leads to the concept of quantum category (while a quantum group considered as a "group" of transformations of a certain quantum space corresponds to a particular choice of deformation). Let us consider a simple example. It is new and serves as a very good illustration of the basic ideas of this work.

Example. Take matrix elements of $T_{\alpha\beta} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as generators of an associative algebra $M_{\alpha\beta}$ and impose the following commutation relations

$$\begin{aligned} ab - \lambda q_\beta^{-1} ba &= 0, \\ ac - \lambda q_\alpha ca &= 0, \\ ad - q_\alpha q_\beta^{-1} da &= (\lambda - \lambda^{-1}) q_\alpha cb \\ bc - q_\alpha q_\beta cb &= 0, \\ bd - \lambda q_\alpha db &= 0, \\ cd - \lambda q_\beta^{-1} dc &= 0, \end{aligned} \tag{0.1}$$

where $\lambda \neq 0, \pm i, \;\; q_\alpha, q_\beta \neq 0$. (We shall treat $T_{\alpha\beta}$ as the matrix of a homomorphism from the quantum deformation of the linear space $\mathbb{C}^{\nvDash}$ with the parameter q_α to that with the parameter q_β. This is explained in the main text below.) The category nature of the transformation $T_{\alpha\beta}$ is reflected in the fact that if we take the matrix $T_{\alpha\beta}$ suiting (1) and the matrix $T_{\beta\gamma}$ suiting (1) with q_β instead of q_α and q_γ instead q_β, then the product $T_{\alpha\gamma} = T_{\alpha\beta} T_{\beta\gamma}$ also suits (1) with q_α, q_γ as the parameters (the matrix elements of the different matrices are assumed to be commuting). If $q_\alpha = q_\beta = \lambda = 1$ then we obtain the usual algebra of functions on the matrix semigroup Mat(2). Thus it is wise to think that after the quantization this semigroup becomes a "quantum category" (with q_α as "objects") and the group GL(2) after the quantization turns a "quantum groupoid" if one adds a formal inverse $(\det_{\alpha\beta}(T_{\alpha\beta}))^{-1}$ to each of the algebras $M_{\alpha\beta}$, where $\det_{\alpha\beta}(T_{\alpha\beta})$ stands for $ad - \lambda q_\alpha cb = ab - \lambda q_\beta^{-1} bc$. The algebras $M_{\alpha\beta}$ possess the "Poincare - Birkhoff - Witt" property: the ordered monomials in matrix entries form the additive bases. For $q_\alpha = q_\beta = 1$ the algebra $M_{\alpha\alpha}$ reduces to the function algebra on the quantum group $\mathrm{GL}(2)_\mathrm{q}$ defined in [26], with $q = \lambda$.

(The appearance of "quantum categories" in such context can be related to the thesis QUANTIZATION ELIMINATES DEGENERACY suggested by Reshetikhin, Takhtajan, Faddeev [26].)

The results of this paper were obtained in November 1994 – January 1995. They were reported at the seminar on quantum groups under guidance of J.Bernstein, J.Donin and S.Shnider in Bar-Ilan University (Ramat-Gan, Israel).

Contents.

The paper is divided into three sections.

In Section 1 we introduce generalized bialgebras which are in the same relation to quantum categories as usual Hopf algebras are to quantum groups.

In Section 2 the quantum deformation of the category of vector superspaces is constructed. We make use of the method of the "universal coacting" known for quantum groups [18],[19],[12],[13]. We calculate the commutational relations and consider various examples including the quantization of the dual space, the quantization of bilinear forms etc.

In Section 3 we present a generalized R-matrix form for the commutational relations on the quantum category $\mathsf{Lin_q}$ and establish a criterion for the classical value of the dimension of the function algebra. This is a technically hard theorem, which may be considered as the main theorem of this paper. The connection with the "Yang – Baxter structures" on quantum linear spaces is established.

In the Appendix we give a general definition of "indexed bialgebras" which include both the bialgebras defined in Section 1 and those dual to them.

Notation.

Throughout the paper the standard tensor notation is used with the regard of $\mathbb{Z}_{\not{2}}$-grading. (For the "super" notions that are used we refer to [6],[8].) For some reason it is convenient to write the coordinates of a vector as a row and those of a covector as a column, and to consider their pairing in the form $\langle v, v'\rangle$, $v \in V$, $v' \in V'$.

Acknowledgements.

I wish to thank V.M.Buchstaber for the discussions of the results of this paper and for supplying the texts of the texts of [12] and [18], and O.M.Khudaverdian for the discussions of the role of categories in connection to mathematical physics. I also wish to thank J.Bernstein, J.Donin and S.Shnider for the discussions in Tel-Aviv and Bar-Ilan universities and for their hospitality in May–June 1995. I thank J.Donin and S.Shnider for giving me their paper [14] and I thank S.Shnider for a copy of his remarkable book with S.Sternberg [27].

1 Generalized bialgebras

We are going to introduce a generalization of bialgebras as to make possible the discussion of "quantized categories", in the same way as the usual bialgebras and Hopf algebras serve as the foundation of the notion of a "quantum group". (More general definition of an "indexed bialgebra" will be discussed in the appendix.)

Consider a set of indices Λ ("objects").

Definition 1.1. A set $A = (A_{\alpha\beta})$, $\alpha, \beta \in \Lambda$ of linear spaces or modules over a commutative ring is said to be a *coalgebra with objects* Λ if linear maps $\Delta_{\alpha\beta\gamma} : A_{\alpha\gamma} \longrightarrow A_{\alpha\beta} \otimes A_{\beta\gamma}$ are given, for all $\alpha, \beta, \gamma \in \Lambda$. It is called a *coassociative coalgebra* if the diagram

$$
\begin{array}{ccc}
A_{\alpha\delta} & \xrightarrow{\Delta_{\alpha\beta\delta}} & A_{\alpha\beta} \otimes A_{\beta\delta} \\
{\scriptstyle \Delta_{\alpha\gamma\delta}}\Big\downarrow & & \Big\downarrow{\scriptstyle 1\otimes\Delta_{\beta\gamma\delta}} \\
A_{\alpha\gamma} \otimes A_{\gamma\delta} & \xrightarrow[\Delta_{\alpha\beta\gamma}\otimes 1]{} & A_{\alpha\beta} \otimes A_{\beta\gamma} \otimes A_{\gamma\delta}
\end{array}
\qquad (1.1)
$$

is commutative for every $\alpha, \beta, \gamma, \delta \in \Lambda$.

A set of linear maps $\varepsilon_\alpha : A_{\alpha\alpha} \longrightarrow R$ (here R is the main field or ring) with the property that all diagrams of the form

$$\begin{tikzcd} A_{\alpha\alpha} \otimes A_{\alpha\beta} \arrow[d, "\varepsilon_\alpha \otimes 1"'] & A_{\alpha\beta} \arrow[l, "\Delta_{\alpha\alpha\beta}"'] \arrow[r, "\Delta_{\alpha\beta\beta}"] \arrow[d] & A_{\alpha\beta} \otimes A_{\beta\beta} \arrow[d, "1 \otimes \varepsilon_\beta"] \\ R \otimes A_{\alpha\beta} \arrow[r] & A_{\alpha\beta} & A_{\alpha\beta} \otimes R \arrow[l] \end{tikzcd} \tag{1.2}$$

must be commutative will be called a *counit* in A. (Here $R \otimes A_{\alpha\beta} \longrightarrow A_{\alpha\beta}$, $A_{\alpha\beta} \otimes R \longrightarrow A_{\alpha\beta}$ are the natural isomorphisms and middle vertical arrow is the identity map.)

The definition of the *homomorphism* of coalgebras $f : (A_{\alpha\beta})_{\alpha,\beta\in\Lambda} \longrightarrow (B_{i,j})_{i,j\in I}$ is obvious. (A map of sets $\alpha \mapsto i(\alpha)$ must be given.)

Definition 1.2. Let all modules be the associative R-algebras with units and all maps $\Delta_{\alpha\beta\gamma}$ and ε_α be the R-linear algebra homomorphisms. Then we shall call A a *bialgebra with objects* Λ.

In the sequel we shall simply say "coalgebra" and "bialgebra" having in mind the given definitions.

Example 1.1. *Functions on a category.* Let C be a small category with the object set $\operatorname{Ob}\mathsf{C} = \Lambda$. Define $A_{\alpha\beta}$ as the function algebra on $\operatorname{Mor}(\beta, \alpha)$. Then the composition induces the comultiplication $\Delta_{\alpha\beta\gamma} : A_{\alpha\gamma} \longrightarrow A_{\alpha\beta} \otimes A_{\beta\gamma}$ and the embedding of the unit $1_\alpha \in \operatorname{Mor}(\alpha, \alpha)$ induces the counit homomorphism $\varepsilon_\alpha : A_{\alpha\alpha} \longrightarrow R$. (We assume the category and the class of functions in consideration to be such that the Cartesian product of the morphism sets corresponds to a tensor product of function algebras.) We obtain the bialgebra with the object set Λ. Here all algebras $A_{\alpha\beta}$ are commutative.

Example 1.2. *Additive categories.* An additive category A can be treated as "ring with several objects" [21]. The additive category axioms are exactly dual (in the sense of reverting arrows) to our definition of a coalgebra (over $\mathbb{Z}$, coassociative and with a counit). An important case is additive categories like $\mathbb{Z}\mathsf{C}$ or $R\mathsf{C}$, generated by the morphisms of an arbitrary small category C (an analogue of the group ring). In this case each module $A_{\alpha\beta} = R\operatorname{Mor}_{\mathsf{C}}(\beta, \alpha)$ is endowed with the comultiplication $A_{\alpha\beta} \longrightarrow A_{\alpha\beta} \otimes A_{\alpha\beta}$, defined on $a_i \in \operatorname{Mor}_{\mathsf{C}}(\beta, \alpha)$ by the formula $\Delta a_i = a_i \otimes \alpha_i$ (the same as for a group ring). Having this additional structure in mind, we may say that the category algebra is a "bialgebra", but not in the sense of the definition above but of the "dual" definition (w.r.t. the reverse arrows). The comultiplication is symmetric here. Considering the dual modules $A'_{\alpha\beta} = \operatorname{Hom}(A_{\alpha\beta}, R)$ we (under certain assumptions) come to a bialgebra. It coincides with the bialgebra of the previous example if, for example, all sets $\operatorname{Mor}(\alpha, \beta)$ are finite.

Example 1.3. "*Supercategories*". They can be considered in the same fashion as Lie supergroups. (For instance one can obviously define the supercategory of $\mathbb{Z}_{\not\simeq}$-graded vector spaces, which contains the general linear supergroups, and in the same way the (infinite-dimensional) supercategory of smooth manifolds.) If for a given supercategory we consider function algebras on the supermanifolds $\mathbf{Mor}(\beta, \alpha)$, then, with standard comments on tensor products, we again obtain a bialgebra with the multiplication commutative in $\mathbb{Z}_{\not\models}$-graded sense.

Thus categories and supercategories can be described by bialgebras with commutative multiplication. It is natural to think that arbitrary bialgebras (in our sense) without commutativity condition may be associated with "quantized categories". Sure, it makes sense in the case a classical (super)category is at hand and the "quantum category" in consideration reduces to it for certain values of the "parameters" (in most broad sense) on which the construction depends.

Definition 1.3. By an *antipode* in a bialgebra $A = (A_{\alpha\beta})$ we shall mean a set of algebra antihomomorphisms $S_{\alpha\beta} : A_{\alpha\beta} \longrightarrow A_{\beta\alpha}$ such that the diagram

$$\begin{tikzcd}
A_{\alpha\beta} & R \arrow[l] \arrow[r] & A_{\beta\alpha} \\
& A_{\alpha\alpha} \arrow[u, "\varepsilon_\alpha"'] \arrow[d, "\Delta_{\alpha\beta\alpha}"] & \\
A_{\alpha\beta} \otimes A_{\alpha\beta} \arrow[uu, "m"] & A_{\alpha\beta} \otimes A_{\beta\alpha} \arrow[l, "1\otimes S_{\alpha\beta}"] \arrow[r, "S_{\beta\alpha}\otimes 1"'] & A_{\beta\alpha} \otimes A_{\beta\alpha} \arrow[uu, "m"']
\end{tikzcd} \tag{1.3}$$

is commutative.

Bialgebras with antipode correspond to "quantum groupoids", i.e. to quantum categories in which "all arrows are invertible".

In classical situation the notion of a "concrete category" is important (that is a subcategory of the category of sets). A bialgebra $(A_{\alpha\beta})$ is called *concrete*, if the "coaction" on a given set of algebras (A_α) is provided. That is the algebra homomorphisms $\delta_{\alpha\beta} : A_\beta \longrightarrow A_\alpha \otimes A_{\alpha\beta}$ that are compatible with the comultiplication: $(\delta_{\alpha\beta} \otimes 1) \circ \delta_{\beta\gamma} = (1 \otimes \Delta_{\alpha\beta\gamma}) \circ \delta_{\alpha\gamma}$ for all α, β, γ.

In a similar way one can transfer to quantum categories the notions of a covariant functor (a representation of a category), a dual category etc. Note that in the following it will be convenient to consider both "left" and "right" coactions. Depending on it, the algebra $A_{\alpha\beta}$ can be interpreted as functions either on the arrows $\alpha \longrightarrow \beta$ or $\alpha \longleftarrow \beta$.

2 The construction of the quantum category Lin_q. Examples.

Let A_α be a set of algebras. Consider the following problem. Consider sets of algebras $A_{\alpha\beta}$, coacting on a given set of algebras Aa, in a sense that algebra homomorphisms $\delta_{\alpha\beta} : A_\beta \longrightarrow A_\alpha \otimes A_{\alpha\beta}$ are given for all indices , and let us look for the universal (initial) object among such coactions. In other words for algebras $\tilde{A}_{\alpha\beta}$ coacting in the same sense on A_α, there is a unique set of homomorphisms $A_{\alpha\beta} \longrightarrow \tilde{A}_{\alpha\beta}$ for which the following diagram

$$
\begin{array}{ccc}
A_\beta & \longrightarrow & A_\alpha \otimes A_{\alpha\beta} \\
& \searrow & \downarrow \\
& & A_\alpha \otimes \tilde{A}_{\alpha\beta}
\end{array}
\tag{2.1}
$$

is commutative.

Theorem 2.1. *The universal set of algebras $A_{\alpha\beta}$ (if exists) is a bialgebra. The comultiplication and counit are canonically defined by the compatibility with the coaction.*

Proof. The iteration of the coaction is always defined, hence for any three indices there exists a homomorphism $A_\gamma \longrightarrow A_\beta \otimes A_{\beta\gamma} \longrightarrow A_\alpha \otimes A_{\alpha\beta} \otimes A_{\beta\gamma}$. From the universality we obtain homomorphisms $A_{\alpha\gamma} \longrightarrow A_{\alpha\beta} \otimes A_{\beta\gamma}$. Similarly, the isomorphism $A_\alpha \longrightarrow A_\alpha \otimes R$ yields a counit homomorphisms $A_{\alpha\alpha} \longrightarrow R$. The coassociativity and the counit property are easily obtained from the uniqueness part of the universality condition.

It is quite obvious that we can similarly treat two sets of algebras A_α, B_α (or any number of them) demanding that $A_{\alpha\beta}$ should coact on both. Or one can put additional constraints on the exact outlook of the coactions. Then the similar universal coacting bialgebra (if exists) will be subject to more tight restrictions. The claim of the theorem still holds.

We shall apply this construction as follows. Consider a set of $\mathbb{Z}_{\not\in}$-graded vector spaces (over a field k). In parallel to each space V we consider the space $V\Pi$, where Π stands for the parity reversion functor. Consider the dual spaces (the spaces of linear functions). A canonical even isomorphism

$$
V' \otimes V' \cong \Pi V' \otimes \Pi V', \tag{2.2}
$$

takes the basis tensors $e^A \otimes e^B$ to $(e^A \otimes e^B)^\Pi := (-1)^{\tilde{A}} (\Pi e^A) \otimes (\Pi e^B)$. Let us fix a decompositipon of $V' \otimes V'$ to two complementary subspaces : $V' \otimes V' = I \oplus J$. Consider the quadratic algebras $(V', I) := T(V')/(I)$ $(\Pi V', J^\Pi) := T(\Pi V')/(J^\Pi)$. (Quotient by the ideals generated by I and J. By $^\Pi$ we denote the isomorphism (2).) We call (V', I) and $(\Pi V', J^\Pi)$ the (polynomial) *function algebras on quantum superspaces* V_q and $V_q\Pi$. Thus the definition of quantum superspace depends on our choice of the decomposition $I \oplus J = V' \otimes V'$, and the spaces V_q and $V_q\Pi$ are defined simultaneously, not independently. Denote by x^A and ξ^A the image of basic linear functions e^A and Πe^A in the algebras (V', I) $(\Pi V', J^\Pi)$ respectively. We shall call them the *coordinates* on quantum superspaces V_q $V_q\Pi$.

Example 2.1. Take I spanned by $e^A \otimes e^B - (-1)^{\tilde{A}\tilde{B}} e^B \otimes e^A$ (the basis of the skew-symmetric tensors), and take J spanned by $e^A \otimes e^B + (-1)^{\tilde{A}\tilde{B}} e^B \otimes e^A$ (the basis of the symmetric tensors). Then the relations in (V', I) and $(\Pi V', J^\Pi)$ are plain commutativity relations: $x^A x^B = (-1)^{\tilde{A}\tilde{B}} x^B x^A$, $\xi^A \xi^B = (-1)^{(\tilde{A}+1)(\tilde{B}+1)} \xi^B \xi^A$ (can be checked). Thus quantum superspaces actually include classic ones.

Do this for each V. We obtain a set of quantum superspaces V_q, $V_q\Pi$, i.e. a set of algebras $(V', I), (\Pi V', J^\Pi)$ which are parametrized by triples: (V, I, J). Denote

$A_\alpha = (V', I), B_\alpha = (\Pi V', J^\Pi)$, where $\alpha = (V, I, J)$, and consider a coaction

$$\begin{aligned} \delta : A_\beta &\longrightarrow A_\alpha \otimes M_{\alpha\beta}, \\ \delta : B_\beta &\longrightarrow B_\alpha \otimes M_{\alpha\beta}. \end{aligned} \tag{2.3}$$

The algebras (V', I) and $(\Pi V', J^\Pi)$ inherit the $\mathbb{Z}$-grading from the tensor algebra. (Not to be confused with the parity.) We demand that the coaction must preserve this grading ("the linearity condition"). Then $\delta(y^K) = x^A \otimes t_A{}^K$, $\delta(\eta^K) = \xi^A \otimes s_A{}^K$, where x^A, y^K, ξ^A, η^K are coordinates on $V_q, W_q, V_q\Pi, W_q\Pi$ respectively, and $t_A{}^K, s_A{}^K$ are certain elements of the algebra $M_{\alpha\beta}$. We also demand that $t_A{}^K = s_A{}^K$ ("compatibily with the functor Π").

Theorem 2.2. *For the set of pairs $(V', I), (\Pi V', J^\Pi)$, with the given constraints (linearity and commuting with Π), there exists the universal coacting. This is a bialgebra $M = (M_{\alpha\beta})$, where all $M_{\alpha\beta}$ are quadratic algebras. The choice of bases in V and W determines the generators $t_A{}^K \in M_{\alpha\beta}$, where $\alpha = (V, I_V, J_V), \beta = (W, I_W, J_W)$, and*

$$\delta(y^K) = x^A \otimes t_A{}^K, \quad \delta(\eta^K) = \xi^A \otimes t_A{}^K. \tag{2.4}$$

The quadratic relations in $M_{\alpha\beta}$ are the following:

$$\begin{aligned} (-1)^{\tilde{B}\tilde{K}} f^{AB}_{(J)} f^{(I)}_{KL} t_A{}^K t_B{}^L = 0, \\ (-1)^{\tilde{B}\tilde{K}} f^{AB}_{(I)} f^{(J)}_{KL} t_A{}^K t_B{}^L = 0, \end{aligned} \tag{2.5}$$

where we denote by $f^{(I)}, f^{(J)}$ the basis tensors in $I \oplus J = V' \otimes V'$ (and the same for W) and by $f_{(I)}, f_{(J)}$ the corresponding elements of the dual basis. The comultiplication $\Delta : M_{\alpha\gamma} \longrightarrow M_{\alpha\beta} \otimes M_{\beta\gamma}$ and the counit $\varepsilon : M_{\alpha\alpha} \longrightarrow k$ are defined by the standard formulas

$$\Delta(t_A{}^S) = t_A{}^K \otimes t_K{}^S, \tag{2.6}$$

$$\varepsilon(t_A{}^B) = \delta_A{}^B \tag{2.7}$$

Proof. We shall show that the relation (2.5) is valid in any coacting $A_{\alpha\beta}$. Any coaction is defined by the formula (3.5), with some $t_A{}^K$. That δ is homomorphic is equivalent to the condition $\bar{\delta}(I_W) \subset I_V \otimes A_{\alpha\beta} \subset (V' \otimes V') \otimes A_{\alpha\beta}$ and $\bar{\delta}(J^\Pi_W) \subset J^\Pi_V \otimes A_{\alpha\beta} \subset (\Pi V' \otimes \Pi V') \otimes A_{\alpha\beta}$, where $\bar{\delta}$ stands for the "covering" coaction on the free tensor algebras, which exists independently of the structure of the algebra $A_{\alpha\beta}$. Consider first the conditions following from the relation (I). To avoid confusion with the elements of the tensor product of two algebras we shall omit the symbol $\otimes$ for the tensors on a given space (so below $e_A e_B := e_A \otimes e_B$ etc). Suppose $f = f_{KL} e^K e^L \in I_W \subset W' \otimes W'$, $\quad g = g^{AB} e_A e_B \in \operatorname{Ann} I_V \subset V \otimes V$. Then for any f and g it is necessary that $0 = \langle g, \bar{\delta}(f) \rangle = g^{AB} f_{KL} \langle e_A e_B, (e^C \otimes t_C{}^K)(e^D \otimes t_D{}^L) \rangle = g^{AB} f_{KL} (-1)^{\tilde{D}(\tilde{C}+\tilde{K})} \langle e_A e_B, e^C e^D \otimes t_C{}^K t_D{}^L \rangle = g^{AB} f_{KL} (-1)^{\tilde{D}(\tilde{C}+\tilde{K})} (-1)^{\tilde{B}\tilde{C}} \delta_A{}^C \delta_B{}^D t_C{}^K t_D{}^L = (-1)^{\tilde{B}\tilde{K}} g^{AB} f_{KL} t_A{}^K t_B{}^L$. Thus the condition $\bar{\delta}(I_W) \subset I_V \otimes A_{\alpha\beta}$ is equivalent to

$$(-1)^{\tilde{B}\tilde{K}} g^{AB} f_{KL} t_A{}^K t_B{}^L = 0, \tag{2.8}$$

where as f the basis elements of I_W can be taken and as g the basis elements of $\operatorname{Ann} I_V$. Now we notice that the condition $\bar{\delta}(J_W^\Pi) \subset J_V^\Pi \otimes A_{\alpha\beta}$ is equivalent to $\bar{\delta}(J_W) \subset J_V \otimes A_{\alpha\beta}$. This follows from the fact that $\bar{\delta}$ commutes with the isomorphism (2.2), i.e. $(\bar{\delta} f)^\Pi = \bar{\delta}(f^\Pi)$ for any $f \in W' \otimes W'$. This immediately follows from the equality $\bar{\delta}(\Pi w') = \Pi \bar{\delta}(w')$ (the Π-symmetry of the coaction). Thus it is proven that in any coacting the relations (2.8) are valid, for $f \in I_W, \quad g \in \operatorname{Ann} I_V$ or $f \in J_W, \quad g \in \operatorname{Ann} J_W$. Taking into account that I and J are complementary one rewrite the relations as (2.5). Now take the matrix entries ${t_A}^K$ as independent variables and consider the associative algebras generated by them with the defining relations (2.8). The set of algebras obtained will be universal. Indeed, for an arbitrary coacting $A_{\alpha\beta}$ change the notation of the matrix elements to ${s_A}^K$, then for any homomorphism commuting with the coaction by necessity ${t_A}^K \mapsto {s_A}^K$. But this formula defines the homomorphism uniquely (since ${t_A}^K$ are generators) and it is well-defined because of the relation proven above. $\square$

Remarks. 1. As follows from the proof, the universal coacting for the pairs of algebras $T(V')/(I), T(\Pi V')/(J^\Pi)$ and for the pairs of algebras $T(V')/(I), T(V')/(J)$ will be the same. Our preference is explained by the fact that in the classical case (see above) both algebras $T(V')/(I)$ and $T(\Pi V')/(J^\Pi)$ are commutative (in $\mathbb{Z}_{\not 2}$-graded sense), while the algebra $T(V')/(J)$ is not commutative (odd generators anticommute with even ones). The same alternative exists for the definition of the exterior algebra, see [2].

2. If for each V we fix a family of complementary subspaces $I_k^V \subset V' \otimes V'$ and look for a universal coaction for the algebras $T(V')/(I_k^V)$ (with the same form of the coaction on the generators, $\bar{\delta}(e^K) = e^A \otimes {t_A}^K$, for all k), then we shall obtain a bialgebra $M = (M_{\alpha\beta})$, with the following commutational relations for each of $M_{\alpha\beta}$ (matrix entries ${t_A}^K$ serve as generators):

$$(-1)^{\tilde{B}\tilde{K}} g_{(k)}^{AB} f_{KL}^{(k)} {t_A}^K {t_B}^L = 0, \tag{2.9}$$

$f^{(k)}$ belongs to the basis of I_k^W, $g_{(k)}$ to the basis of $\operatorname{Ann} I_k^V$. Here the "numbers" of objects are $\alpha = (V, I_1^V, \ldots, I_s^V)$ and $\beta = (W, I_1^W, \ldots, I_s^W)$.

3. The relations (2.5) are linear independent. It easy to calculate that for the classical values of $\dim I_V$, $\dim J_V$, $\dim I_W$, $\dim J_V$ the dimension of the quadratic part of the "quantum algebra" $M_{\alpha\beta}$ will be classical too. The question whether the same will be true for the higher order terms is highly non-trivial. We discuss it in the next section.

4. In quantum group theory the universal coaction method was proposed by Yu.A.Kobyzev (see [18]). Originally a single quadratic algebra $A = T(V')/(I)$ was considered. This gives only "half" of the necessary commutational relations on quantum (semi)group. This difficulty has been resolved in a rather artificial manner by using the "dual" quadratic algebra $A^!$ (see the remark after Example 2.5 below). In particular that implied that a single subspace $I \subset V' \otimes V'$ played the role of the "quantization parameter". The application of families of complementary subspaces is described in [12], in purely even case. (The functor Π is not used in [12].) It is worth noting that in $\underline{\hom}(A,B) = A^! \bullet B$ was introduced for each pair of quadratic algebras A and B, together with a sort of "coproduct" $\Delta : \underline{\hom}(A,C) \longrightarrow \underline{\hom}(B,C) \circ \underline{\hom}(A,B)$ (we refer also to [9] for the notation). The algebras $\underline{\hom}$ are not so interesting (they, and in particular $\underline{\mathrm{end}}(A) = \underline{\hom}(A,A)$, indeed lack half of the relations, so they are strongly non-commutative). But having in mind the natural homomorphism $A \circ B \longrightarrow A \otimes B$ category in our sense, or. more precisely, an examlple of the generalized bialgebra (in the sense of Definition 1.2). The interpetation given in [18] was completely different. It was based on the "internal Hom" formalism of the "rigid tensor categories", which are just actual categories endowed with the additional structure.

Definition 2.1. The bialgebra $M = (M_{\alpha\beta})$, defined in Theorem 2.2, will be called the *algebra of* (polynomial) *functions on the quantum category* Lin_q.

Example 2.2. Consider classical (not deformed) spaces. For every V the subspaces $\mathrm{Ann}\, I, \mathrm{Ann}\, J \subset V \otimes V$ are spanned by the tensors $e^A \otimes e^B + (-1)^{\tilde{A}\tilde{B}} e^B \otimes e^A$ and $e^A \otimes e^B - (-1)^{\tilde{A}\tilde{B}} e^B \otimes e^A$. By Theorem 2.2 we obtain the following relations for the matrix elements of the homomorphism from V to W:

$$(\delta_A{}^{A'}\delta_B{}^{B'} + (-1)^{\tilde{A}\tilde{B}}\delta_B{}^{A'}\delta_A{}^{B'})(\delta_{K'}{}^K\delta_{L'}{}^L - (-1)^{\tilde{K}\tilde{L}}\delta_{K'}{}^L\delta_{L'}{}^K)(-1)^{\tilde{B}'\tilde{K}'}t_{A'}{}^{K'}t_{B'}{}^{L'} = 0,$$
$$(\delta_A{}^{A'}\delta_B{}^{B'} - (-1)^{\tilde{A}\tilde{B}}\delta_B{}^{A'}\delta_A{}^{B'})(\delta_{K'}{}^K\delta_{L'}{}^L + (-1)^{\tilde{K}\tilde{L}}\delta_{K'}{}^L\delta_{L'}{}^K)(-1)^{\tilde{B}'\tilde{K}'}t_{A'}{}^{K'}t_{B'}{}^{L'} = 0.$$

Then, after summation, we obtain:

$$(-1)^{\tilde{B}\tilde{K}}t_A{}^K t_B{}^L - (-1)^{\tilde{K}\tilde{L}+\tilde{B}\tilde{L}}t_A{}^L t_B{}^K + (-1)^{\tilde{A}\tilde{B}+\tilde{A}\tilde{K}}t_B{}^K t_A{}^L - (-1)^{\tilde{A}\tilde{B}+\tilde{K}\tilde{L}+\tilde{A}\tilde{L}}t_B{}^L t_A{}^K = 0$$

and a similar equality with the opposite signs before the second and the third terms. Summing and substracting these equalities, we get

$$t_A{}^K t_B{}^L - (-1)^{(\tilde{A}+\tilde{K})(\tilde{B}+\tilde{L})}t_B{}^L t_A{}^K = 0,$$

for any A, B, K, L, i.e. the usual commutativity condition.

Example 2.3. Take one-dimensional k as W with 0 and $k \cong k \otimes k$ as I_W, J_W respectively. Then by Theorem 2.2 we get

$$g_{(J)}^{AB} t_A t_B = 0 \tag{2.10}$$

as commutational relations for the coefficients of even *linear forms* on $V_q, V_q\Pi$. Here $g_{(J)} \in \mathrm{Ann}\, J_V$. In the same way, for the coefficients of *odd linear forms* θ_A, $\quad \tilde{\theta}_A = \tilde{A}+1$, we obtain the relations

$$(-1)^{\tilde{B}} g_{(I)}^{AB} \theta_A \theta_B = 0, \tag{2.11}$$

where $g_{(I)} \in \mathrm{Ann}\, I_V$. In other words, if one identifies a quantum space with a triple (V, I, J), then its *quantum dual space* will be $(V', \mathrm{Ann}\, J, \mathrm{Ann}\, I)$. Pairings: $\langle x, t\rangle = x^A \otimes t_A$, $\quad \langle x, \theta\rangle = x^A \otimes \theta_A$.

Example 2.4. From the previous example one can deduce the commutational relations for the coefficients of (even) *bilinear forms*, considered as homomorphisms to dual space ("lowering indices"):

$$\begin{aligned}(-1)^{\tilde{B}\tilde{C}} g_{(I)}^{AC} g_{(J)}^{BD} t_{AB} t_{CD} = 0,\\ (-1)^{\tilde{B}\tilde{C}} g_{(J)}^{AC} g_{(I)}^{BD} t_{AB} t_{CD} = 0,\end{aligned} \tag{2.12}$$

where $\tilde{t}_{AB} = \tilde{A} + \tilde{B}$, $\quad g_{(I)}, g_{(J)}$ are tensors in $\operatorname{Ann} I, \operatorname{Ann} J \subset V \otimes V$ respectively. The bilinear forms are: $\langle x_1 \mid T \mid x_2 \rangle = x_1^A \otimes t_{AB} \otimes x_2^B$ and the similar for ξ_i.

Example 2.5. Consider the quantum spaces with the following commutational relations:

$$\begin{aligned}x^A x^B - q^{AB} x^B x^A = 0,\\ \xi^A \xi^B + (-1)^{\tilde{A}+\tilde{B}} p^{AB} \xi^B \xi^A = 0.\end{aligned} \tag{2.13}$$

Here $q^{AB} = (q^{BA})^{-1}$, $p^{AB} = (p^{BA})^{-1}$, $q^{AA} = p^{AA} = (-1)^{\tilde{A}}$. (In the "classical limit" $q^{AB}, p^{AB} \longrightarrow (-1)^{\tilde{A}\tilde{B}}$.) Look for the relations in the function algebra for the corresponding "full subcategory" of Lin_q. Here the objects are triples (V, P_V, Q_V), where $P_V = (p^{AB}) = (p_{(V)}^{AB})$, $Q_V = (q^{AB}) = (q_{(V)}^{AB})$ are the matrices of the parameters. The subspace I is spanned by the tensors $e^A \otimes e^B - q^{AB} e^B \otimes e^A$, J by the tensors $e^A \otimes e^B + p^{AB} e^B \otimes e^A$, $\operatorname{Ann} I$ by the tensors $e_A \otimes e_B + q_{BA} e_B \otimes e_A$, and $\operatorname{Ann} J$ by the tensors $e_A \otimes e_B - p_{BA} e_B \otimes e_A$. As not to contradict the tensor notation, we have introduced here the parameters with the lower indices: $q_{AB} := q^{AB}$, $\quad p_{AB} := p^{AB}$. The complementarity of I and J is equivalent to

$$q^{AB} + p^{AB} \neq 0, \tag{2.14}$$

(for any V). Similarly to Example 2.1 we obtain two relations for the matrix elements of a "linear map of the quantum space (V, P_V, Q_V) to the quantum space (W, P_W, Q_W)":

$$(-1)^{\tilde{B}\tilde{K}} {t_A}^K {t_B}^L - q^{KL} (-1)^{\tilde{B}\tilde{L}} {t_A}^L {t_B}^K + q_{BA}((-1)^{\tilde{A}\tilde{K}} {t_B}^K {t_A}^L - q^{KL} (-1)^{\tilde{A}\tilde{L}} {t_B}^L {t_A}^K) = 0$$
$$(-1)^{\tilde{B}\tilde{K}} {t_A}^K {t_B}^L + p^{KL} (-1)^{\tilde{B}\tilde{L}} {t_A}^L {t_B}^K - p_{BA}((-1)^{\tilde{A}\tilde{K}} {t_B}^K {t_A}^L + p^{KL} (-1)^{\tilde{A}\tilde{L}} {t_B}^L {t_A}^K) = 0$$

Here $Q_V = (q^{AB}), P_V = (p^{AB}), Q_W = (q^{KL}), P_W = (p^{KL})$. Provided 2.14, these defining relations can be identically transformed to the following final form:

$$\begin{aligned}{t_A}^K {t_B}^L - \frac{p_{BA} + q_{BA}}{p^{LK} + q^{LK}} (-1)^{\tilde{A}\tilde{L} + \tilde{B}\tilde{K}} {t_B}^L {t_A}^K = \\ \frac{p_{BA} p^{LK} - q_{BA} q^{LK}}{p^{LK} + q^{LK}} (-1)^{(\tilde{A}+\tilde{B})\tilde{K}} {t_B}^K {t_A}^L\end{aligned} \tag{2.15}$$

for any A, B, K, L. Notice that for the elements of one column or of one row these relations reduce to

$${t_A}^K {t_B}^K - \frac{p_{BA}(1 + (-1)^{\tilde{K}}) + q_{BA}(1 - (-1)^{\tilde{K}})}{2(-1)^{(\tilde{A}+\tilde{B}+1)\tilde{K}}} {t_B}^K {t_A}^K = 0, \tag{2.16}$$

$${t_A}^K {t_A}^L - \frac{2(-1)^{(\tilde{A})(\tilde{K}+\tilde{L}+1)}}{p^{LK}(1 - (-1)^{\tilde{A}}) + q^{LK}(1 + (-1)^{\tilde{A}})} {t_A}^L {t_A}^K = 0. \tag{2.17}$$

Remarks. 1. The particular case of (2.15) are the relations in the function algebra on the "multiparameter deformation" of the general linear supergroup. The original relations of the form (2.13) for quantum linear spaces (in the even case) together with the relevant quantization of $\mathrm{GL}\,(n)$ is due to Sudbery [28] (see the surveys [12], [13]). Before Manin [19] introduced a multiparameter quantization for supergroup $\mathrm{GL}\,(n \mid m)$ starting from the relations similar to (2.13) with $(q^{AB})^{-1}$ instead of p^{AB} (in our notation). There was a certain confusion in the papers [18], and as a consequence the analogue of the second equation of (2.13) was associated with the "dual quadratic algebra" $A^!$ while the coaction was taken as if it had been for the initial space.

The structure of the formulas (2.15-2.17) is similar to the relations for the standard (single-parameter) deformation of the group $\mathrm{GL}\,(2)$ [26]. The commutational relations for the multiparameter deformations of $\mathrm{GL}\,(n \mid m)$ are commonly presented in a more cumbersome form

2. In the paper [11] Demidov defined for two spaces V and W an algebra close to that defined above, for $p^{AB} = (q^{AB})^{-1}, \quad p^{KL} = (q^{KL})^{-1}$, see the previous remark. But he never considered a comultiplication except for the case of a single space with fixed parameters q^{AB} (i.e. for "quantum semigroups").

Example 2.6. For quantum space with the relations (2.13) the commutational relations for the dual space (see Example 2.3) are:

$$\begin{gathered} t_A t_B - (p_{AB})^{-1} t_B t_A = 0, \\ \theta_A \theta_B - (-1)^{\tilde{A}+\tilde{B}} (q_{AB})^{-1} \theta_B \theta_A = 0. \end{gathered} \tag{2.18}$$

Example 2.7. Similarly for even bilinear forms (see Example 2.4):

$$\begin{multline} t_{AB} t_{CD} - \frac{p_{CA} + q_{CA}}{q_{BD} + p_{BD}} (-1)^{\tilde{A}\tilde{D}+\tilde{B}\tilde{C}} t_{CD} t_{AB} = \\ \frac{p_{CA} q_{BD} - q_{CA} p_{BD}}{q_{BD} + p_{BD}} (-1)^{(\tilde{A}+\tilde{C})\tilde{B}} t_{CB} t_{AD} \end{multline} \tag{2.19}$$

For convenience we shall also write down the relations (2.13–2.17) for the subcategory of purely even spaces (V is purely even, $V\Pi$ is purely odd):

$$\begin{gathered} x^a x^b - q^{AB} x^b x^a = 0, \\ \xi^a \xi^b + p^{AB} \xi^b \xi^a = 0, \end{gathered} \tag{2.20}$$

$$t_a{}^k t_b{}^l - \frac{p_{ba} + q_{ba}}{p^{lk} + q^{lk}} t_b{}^l t_a{}^k = \frac{p_{ba} p^{lk} - q_{ba} q^{lk}}{p^{lk} + q^{lk}} t_b{}^k t_a{}^l, \tag{2.21}$$

$$t_a{}^k t_b{}^k - p_{ba} t_b{}^k t_a{}^k = 0, \tag{2.22}$$

$$t_a{}^k t_a{}^l - (q^{lk})^{-1} t_a{}^l t_a{}^k = 0, \tag{2.23}$$

Definition 2.1 makes sense for just a single vector space V. Here we obtain the "full subcategory" of Lin_q consisting of all linear transformations between various quantum deformations of V defined by the decomposition $I \oplus J = V' \otimes V'$. This quantum category is the adequate quantization of the semigroup $\operatorname{End} V$. We see that the quantum categories language is more to the point here. It is more flexible than the language of quantum (semi)groups.

Example 2.8. Fix a space V, $\dim V = 2$ and let us change the quantization parameters in (2.20). Suppose $p^{21} = p$, $q^{21} = q$,$p = p_\alpha$, $q = q_\alpha$. We arrive to a quantum category, with indices α, β that number the parameters p and q as objects. The commutational relations for the matrix elements of a "homomorphism from (V, α) to (V, β)" in the natural notation are:

$$ab - q_\beta^{-1} ba = 0, \tag{2.24}$$

$$ac - p_\alpha ca = 0, \tag{2.25}$$

$$ad - \frac{p_\alpha + q_\alpha}{p_\beta + q_\beta} da = \frac{p_\alpha p_\beta - q_\alpha q_\beta}{p_\beta + q_\beta} cb, \tag{2.26}$$

$$bc - \frac{p_\alpha + q_\alpha}{p_\beta + q_\beta} p_\beta q_\beta cb = \frac{p_\alpha q_\beta - q_\alpha p_\beta}{p_\beta + q_\beta} da, \tag{2.27}$$

$$bd - p_\alpha db = 0, \tag{2.28}$$

$$cd - q_\beta^{-1} dc = 0. \tag{2.29}$$

From (2.26) and (2.27) the relation between b and c can be expressed also as

$$bc - \frac{p_\beta + q_\beta}{p_\alpha + q_\alpha} p_\alpha q_\alpha cb = \frac{p_\alpha q_\beta - q_\alpha p_\beta}{p_\alpha + q_\alpha} ad. \tag{2.30}$$

For $\alpha = \beta$ these relations reduce to the commutational relations on the quantum group $\mathrm{GL}(2)$.

Example 2.9 (*the determinant on a quantum category*). In the situation of the previous example consider the "area form" $\xi^1 \xi^2$. Substituting $\xi^k_\beta = \xi^a_\alpha t_a{}^k$, we obtain

$$\delta(\xi^1_\beta \xi^2_\beta) = \xi^1_\alpha \xi^2_\alpha \otimes \det_{\alpha\beta}(T). \tag{2.31}$$

The factor denoted by $\det_{\alpha\beta}(T)$ is called, by definition, the *determinant* of the matrix $T = (t_a{}^k)$. Here ξ^a_α are the coordinates on $(\Pi V, \alpha)$ and ξ^k_β the coordinates on $(\Pi V, \beta)$. From the definition and the commutational relations we obtain the equivalent formulas:

$$\begin{aligned}
\det_{\alpha\beta}(T) &= ad - p_\alpha cb \\
&= \frac{p_\alpha + q_\alpha}{p_\beta + q_\beta}(da - q_\beta cb) \\
&= \frac{1 + p_\alpha q_\alpha^{-1}}{p_\beta + q_\beta}(q_\beta ad - bc) \\
&= p_\beta^{-1}(p_\alpha da - bc).
\end{aligned} \tag{2.32}$$

For any "morphisms $\alpha \longrightarrow \beta$, $\beta \longrightarrow \gamma$" the identity

$$\det_{\alpha\gamma}(T_{\alpha\gamma}) = \det_{\alpha\beta}(T_{\alpha\beta}) \det_{\beta\gamma}(T_{\beta\gamma}), \tag{2.33}$$

holds. Here $T_{\alpha\gamma} = T_{\alpha\beta}T_{\beta\gamma}$. Thus det defines a one-dimensional representation of the quantum category in consideration. Mind that the function $\det_{\alpha\beta}$ is defined not uniquely but up to the "adding of a coboundary", i.e. up to a factor $f_\alpha f_\beta^{-1}$, where f_α is a non-vanishing scalar function of the parameters p_α, q_α. (Due to the non-uniqueness of the basis form $\xi^1_\alpha \xi^2_\alpha$). The subcategory with det cohomologous to zero is a correct quantum analogue of the group SL.

3 The Poincare – Birkhoff – Witt property. R-matrix formulation and the connection with the Yang – Baxter equation.

Consider the quantum category Lin_q with triples $(V, I.J)$, $I \oplus J = V' \otimes V'$ as objects. It is possible to consider even more general case of the quantum category with objects like $(V, I_1, \ldots, I_s)$, $\bigoplus I_k = V' \otimes V'$, the number s is fixed (see the remark after Theorem 2.2). Denote it by $\mathsf{Lin}_q^{(s)}$, $s \geqslant 2$. Then $\mathsf{Lin}_q = \mathsf{Lin}_q^{(2)}$.

Theorem 3.1. *Let the matrix entries of $T_{\alpha\beta} = ({t_A}^K)$ generate the algebra $M_{\alpha\beta}$, which is the function algebra on homomorphisms from $\alpha = (V, I_1^V, \ldots, I_s^V)$ to $\beta = (W, I_1^W, \ldots, J_s^W)$, in the quantum category $\mathsf{Lin}_q^{(s)}$. Let $T^1_{\alpha\beta} := T_{\alpha\beta} \otimes 1$, $T^2_{\alpha\beta} := 1 \otimes T_{\alpha\beta}$. Then the defining commutational relations for functions on $\mathsf{Lin}_q^{(s)}$ can be presented in the following "R-matrix form":*

$$B_\alpha(T^1_{\alpha\beta}T^2_{\alpha\beta}) = (T^1_{\alpha\beta}T^2_{\alpha\beta})B_\beta, \tag{3.1}$$

where for each α the matrix B_α is the linear combination of the projectors on the subspaces I_k along the sum $\bigoplus\limits_{l \neq k} I_l$, where the coefficients λ_k are pairwise different and independent on α.

Proof. Consider the partitions of unity $1 = \sum P_k^V$, $1 = \sum P_k^W$ by the projectors corresponding to the direct sum decompositions $V' \otimes V' = \bigoplus I_k^V$, $W' \otimes W' = \bigoplus I_k^W$. Let $A_{\alpha,k} = T(V)/(I_k^V)$, $A_{\beta,k} = T(W)/(I_k^W)$. The homorphisms $\delta : A_{\beta,k} \longrightarrow A_{\alpha,k} \otimes M_{\alpha\beta}$, $k = 1, \ldots, s$, (the coaction) are covered by the map of the tensor algebras $\bar{\delta}$, and the commutational relations in $M_{\alpha\beta}$ are defined by the condition $\bar{\delta}(I_k^W) \subset I_k^V \otimes M_{\alpha\beta}$ for all k. Any linear map $A : W' \otimes W' \longrightarrow V' \otimes V' \otimes M_{\alpha\beta}$ can be uniquely decomposed as $A = \sum P_i^V A P_j^W = \sum A_{ij}$. The condition above is equivalent to that the operator $\bar{\delta}$ is "diagonal" on the tensor square: $\bar{\delta} = \sum \bar{\delta}_{jj}$. Let us introduce $B_\alpha = \sum \lambda_k P_k^V$, $B_\beta = \sum \lambda_k P_k^W$, where $\lambda_k \neq \lambda_j$, $k \neq j$. Then the difference of the left- and right-hand sides of is $B_\beta \circ \bar{\delta} - \bar{\delta} \circ B_\alpha = \sum\limits_{k,j} (\lambda_k - \lambda_j)\bar{\delta}_{kj}$, which vanishes if and only if $\bar{\delta}_{kj} = 0$ for $k \neq j$. That is just the condition defining the commutational relations in $M_{\alpha\beta}$. $\square$

In the index notation the relation (3.1) can be rewritten as follows:

$${t_A}^N {t_B}^M {B_{NM}}^{KL} = {B_{AB}}^{CD} {t_C}^K {t_D}^L, \tag{3.2}$$

where $B_\alpha = ({B_{AB}}^{CD})$, $B_\beta = ({B_{NM}}^{KL})$.

The claim of the theorem is analoguous to the corresponding result for quantum groups, where the relations are determined by a single matrix B.

For any quantum space $(V, I_1, \dots, I_s)$ the projectors P_k can be expressed from the matrix B_α ("R-matrix") as the projections on its eigenspaces. The commutational relations in the algebras $A_k = T(V')/(I_k)$ are also expressed in terms of the operator B_α. Suppose one has the commutational relations in the R-matrix form. It is common then to relate "the Poincare – Birkhoff – Witt property" (the equality of the graded dimension of function algebra on quantum space or quantum group to that of the corresponding classical polynomial algebra) with the identity

$$B^{12}B^{23}B^{12} = B^{23}B^{12}B^{23} \tag{3.3}$$

for the matrix B, where $B^{12} = B \otimes 1, B^{23} = 1 \otimes B$ (the Yang – Baxter equation). Strictly speaking, no general statement exists here and one can only say that a certain "relation" takes place. (As Joseph Donin told me after this work was completed, a sort of theory can be actually developped dealing with the quantum algebra dimension in connection with the YB equation. This is the topic of the recent paper [14] by him and Steve Shnider.)

Consider the case of $\mathsf{Lin}_q = \mathsf{Lin}_q^{(2)}$, $I_1 = I, I_2 = J$. We shall normalize the matrices B so as $B = P_1 - \lambda P_2, \quad \lambda \neq -1$, and we shall look for λ suiting (3.3). In general this is an overdetermined problem. One can notice (see, for example, [12]), that if the solutions exist there may be either exactly two of them: $B_+ = P_1 - \lambda P_2$ and $B_- = P_1 - \lambda^{-1}P_2$, where $\lambda \neq 1$, or just one: $B = P_1 - P_2$, in the case the equation (3.3) is satisfied for $\lambda = 1$. (In the last case an additional identity holds, $B^2 = 1$.) Applying this to our situation, we obtain that if for two quantum spaces $\alpha = (V, I_1^V, I_2^V)$ and $\beta = (W, I_1^W, I_2^W)$ the normalized matrices B_α and B_β suit the Yang – Baxter equation, and we fix the choice of B_α and B_β, then the commutational relations in $M_{\alpha\beta}$ can be presented in the "R-matrix form"(3.1),(3.2) with the given B_α and B_β if and only if $\lambda_\alpha = \lambda_\beta$ or $\lambda_\alpha = \lambda_\beta^{-1}$.

Let us turn now to problem of the graded dimension of the algebras $M_{\alpha\beta}$. We consider the subcategory with defining commutational relations for quantum spaces of the form (2.13). Let $\alpha = (V, P_V, Q_V), \quad \beta = (W, P_W, Q_W)$ with the matrices of parameters $P_V = (p^{AB}), Q_V = (q^{AB}), P_W = (p^{KL}), Q_W = (q^{KL})$. For arbitrary orderings of the tensor indices in V and W let $\mathrm{sign}(B - A)$ equal $+1, 0, -1$ if $B > A,\ B = A,\ B < A$ respectively. In a similar way we define the function $\mathrm{sign}(L - K)$. The matrix entries will be ordered by rows: ${t_A}^K < {t_B}^L$, if $A < B$ or if $A = B, \quad K < L$. The *ordered monomial* is defined as usual to be such that the sequence of letters is not decreasing and any odd variable can appear no more than once.

Theorem 3.2. *The algebra $M_{\alpha\beta}$ is of classical dimension iff the bases in V and W can be ordered in such a way that the following equations hold*

$$p^{AB} = q^{AB} c_\alpha^{\mathrm{sign}(B-A)}, \tag{3.4}$$

$$p^{KL} = q^{KL} c_\beta^{\mathrm{sign}(L-K)}, \tag{3.5}$$

for some constants c_α, c_β, not equal to 0 *or* -1*, and either*

$$c_\alpha = c_\beta, \tag{3.6}$$

or

$$c_\alpha = c_\beta^{-1}, \tag{3.7}$$

In this case the ordered monomials in matrix entries span the algebra $M_{\alpha\beta}$.

Proof. It actually consists of two parts.

First choose an arbitrary ordering of the tensor indices. We shall show that the ordered monomials span the whole algebra $M_{\alpha\beta}$ for any values of the parameters $p^{AB}, q^{AB}, p^{KL}, q^{KL}$. Indeed, consider an arbitrary monomial in ${t_A}^K$. Consider the elements of the first row in it. Let τ stand for the number of the inversions with the elements of other rows. That is the number of cases when the elements of the first row appear to the right of the elements of other rows. Take any neighbouring pair of elements of the sort ${t_A}^K {t_1}^L, A > 1$. By the commutational relations (2.15) it is possible to substitute it by a linear combination of the products ${t_1}^L {t_A}^K \;\; {t_1}^K {t_A}^L$, and the number τ will decrease by one. Thus a finite iteration of this procedure leads to a linear combination of the monomials such that in each of them the elements of the first row stand to the left of the product of the elements of the other rows. We can apply the same method to these products, now taking the second row instead of the first, etc. (If at some step the elements of a certain row are absent, then we take the next row.) As there are finite number of rows, after a finite steps we shall obtain a linear combination of monomials with the correct order of the row numbers. The elements inside the same row (they all are neighbours now) are put in the correct order by the formulas (2.17). Now the arbitrary monomial from which we have started is expressed as linear combination of the ordered mononials.

Such algorithm for a quantum deformation of the supergroup $\mathrm{GL}\,(n \mid m)$ was described by Manin

The second part of the proof deals with the linear independence of the ordered monomials. This is much more non-trivial problem. The so-called “diamond lemma” of the combinatorial theory of rings (see [17], [13]) implies that for quadratic algebras it is necessary and sufficient to check the independence only for the monomials of the third order. (The lemma provides in this case the linear independence also for all ordered monomials of the degree $\geqslant 3$.) Let us look for the criterion of the independence for ordered cubic monomials in ${t_A}^K$.

The linear independence of ordered monomials is equivalent to the uniqueness of the “normal form” of any arbitrary monomial, i.e. to the independence “on path” of the result of its expression as a combination of the ordered monomials. We need to finger all cubic monomials with the broken order and for each of them to compare all the possible ways to put them in the normal order. Let $a < b < c$ be the arbitrary letters symbolizing the matrix entries. Then the cubic monomials with the broken order are the following: acb, bac, bca, cab, cba (three letters different) and aba, ba^2 (two letters different). In the process of putting them into order the new letters can arise. Which ones (together with the ways of putting into order) actually depends on the position of the elements a, b, c in the matrix $(t_A{}^K)$. It is most convenient to draw pictures on which the matrix entries are shown as the vertices of an orthogonal lattice. By (2.15) the "commutator closure" of the given set of letters consists of the vertices of the least sublattice containing this set. Other letters cannot appear in the process of putting the monomial into normal form. For three pairwise different entries we obtain one picture with six additional letters (if the initial elements stand on three different rows) and two pictures with three additional letters (if initially we have elements in two different rows). For two different letters a picture with two additional letters appears. (The pictures with the initial letters in one row do not include any additional letters and they cannot produce any ambiguity. The staightforward checking of all possible ways leading to the sum of ordered monomials in each of these $5 \times 3 + 2 = 17$ cases shows that the non-uniqueness of the normal form is apriori possible only for the monomials like cba (for all three variants of the position of the initial elements), i.e. when branching of the algorithm takes place at the very first step. (Have in mind that the conclusion is strongly based on the exact form of the commutational relation.) Happily no branching is possible here at the following steps. In the most complex case with six additional letters two branches of the algorithm look as follows: $cba \longrightarrow (bca, xya) \longrightarrow (bac, buv, xay, xtv) \longrightarrow (abc, tsc, ubv, txv, axy, usy, txv, ubv)$ and $cba \longrightarrow (cab, cts) \longrightarrow (acb, uvb, tcs, uys) \longrightarrow (abc, axy, ubv, usy, tsc, txv, usy, ubv)$. Here all the arising monomials are successively written down. In two other cases the branches are a bit shorter. The final list of the ordered monomials is the same for both branches. That means that the uniqueness of the normal form is equivalent to the agreement of numeric factors in the similar terms. The explicit calculation leads us to the 12 equations for the parameters $p^{AB}, q^{AB}, p^{KL}, q^{KL}$. Five of these equations happen to be identities. Let us give as example two of the non-trivial equations:

$$\frac{(p_{AB}q^{LN} - q_{AB}p^{LN})(p_{AC}p^{KN} - q_{AC}q^{KN})(p_{BC} + q_{BC})}{(p^{KN} + q^{KN})(p^{LN} + q^{LN})} + \frac{(p_{AB} + q_{AB})(p_{AC}p^{KL} - q_{AC}q^{KL})(p_{BC}p^{LN} - q_{BC}q^{LN})}{(p^{KL} + q^{KL})(p^{LN} + q^{LN})} = \frac{(p_{AB}p^{KL} - q_{AB}q^{KL})(p_{AC} + q_{AC})(p_{BC}p^{KN} - q_{BC}q^{KN})}{(p^{KL} + q^{KL})(p^{KN} + q^{KN})}, \quad (3.8)$$

if $A < B < C$, and

$$\frac{2(p_{AB} + q_{AB})(p_{AB}p^{KN} - q_{AB}q^{KN})}{(p^{KN} + q^{KN})(p^{NL} + q^{NL})(p^{LN}(1 - (-1)^{\tilde{B}}) + q^{LN}(1 + (-1)^{\tilde{B}}))} = \frac{2(p_{AB} + q_{AB})(p_{AB}p^{KN} - q_{AB}q^{KN})}{(p^{KN} + q^{KN})(p^{KL} + q^{KL})(p^{LK}(1 - (-1)^{\tilde{B}}) + q^{LK}(1 + (-1)^{\tilde{B}}))} + \frac{(p_{AB}p^{KL} - q_{AB}q^{KL})(p_{AB}p^{LN} - q_{AB}q^{LN})}{(p^{KL} + q^{KL})(p^{LN} + q^{LN})}, \quad (3.9)$$

if $A < B, L < N$. To get the solution we shall make a reduction. Substitute $K = N$ into (3.9). Two terms cancel immediately and we arrive to $(p_{AB}p^{NL} -$

$q_{AB}q^{NL})(p_{AB}p^{LN} - q_{AB}q^{LN}) = 0$, which is equivalent to

$$\frac{p_{AB}}{q_{AB}} = \frac{p^{LN}}{q^{LN}} \quad \text{or} \quad \frac{p_{AB}}{q_{AB}} = \frac{q^{LN}}{p^{LN}} \tag{3.10}$$

(recall that $p^{LN} = (p^{NL})^{-1}, q^{LN} = (q^{NL})^{-1}$). Let $p_{AB}/q_{AB} = c_{AB}, p_{KL}/q_{KL} = c_{KL}$. The equations (3.10) are equivalent to

$$c_{AB} + (c_{AB})^{-1} = c_{KL} + (c_{KL})^{-1} \tag{3.11}$$

for any $A < B, \quad K < L$. That means that both the right-hand side and the left-hand side of (3.11) do not depend on indices and are equal to some constant $\mu \neq 0$. Thus for all $A, B \quad c_{AB} = c^{\varepsilon_{AB}}$, where $c : c + c^{-1} = \mu$, $\varepsilon_{AB} + \varepsilon_{BA} = 0$, $\varepsilon_{AB} = \pm 1$ if $A \neq B$, and the same for $c_{KL} = c^{\varepsilon_{KL}}$. Let $c \neq 1$. If the matrices $(\varepsilon_{AB}), (\varepsilon_{KL})$ do not possess the "transitivity" property ($\varepsilon_{AB} = 1, \varepsilon_{BC} = 1$ implies $\varepsilon_{AC} = 1$), then the equations are not satisfied. Indeed, substituting, for example, $p_{AB} = c \cdot q_{AB}$, $p_{BC} = c \cdot q_{BC}$, $p_{AC} = c^{-1} \cdot q_{AC}$ into (3.8) and setting $K = L = N$, we obtain a contradiction (for $c \neq 1$). It follows that if it is not true that $\varepsilon_{AB} = \mathrm{sign}(B - A), \varepsilon_{KL} = \mathrm{sign}(L - K)$ for some ordering of the bases of V and W (perhaps different from the originally chosen), then the dimension of the algebra $M_{\alpha\beta}$ is less than the classical value. (The ordered cubic monomials, w.r.t. the original ordering, will be linear dependent and according to what was proven above they actually span the subspace of all cubic functions.) Assume now that this is true. We write the same equations on the deformation parameters w.r.t. the new ordering. To finish the proof it is necessary to substitute in them the equations check shows that all the seven equations are actually satisfied. The case of $c = 1$ is also contained here; it is indifferent to the choice of ordering. $\square$

We actually proved that it is enough to consider the case (3.6), for a suitable ordering. From the other point, it was known that for a single space $\alpha = (V, I_1, I_2)$ the possibility to choose the order of the basis in such a way that the equation (3.4) holds for a certain c_α is equivalent both to the PBW property for the algebra $M_{\alpha\alpha}$ and to the Yang – Baxter equation for $B_\pm = P_1 - (c_\alpha)^{\pm 1} P_2$ (see [12], [13]). The key point of our result is that the equality of "quantum constants" $c_\alpha = c_\beta^{\pm 1}$ for two spaces is exactly the criterion for the classical value of the dimension of the algebra $M_{\alpha\beta}$ (the "Poincare – Birkhoff – Witt" for $M_{\alpha\beta}$). This criterion is in excellent agreement with our heuristic consideration above, which was based on the Yang – Baxter equation, while there is no direct logical connection. For $c_\alpha \neq 1$ one can choose one of the two orderings compatible with (3.4) (direct or reverse), i.e. to choose one of the two "Yang–Baxter structures" B_+ or B_- on (V, I_1, I_2), which could be viewed as a sort of "orientation". The statement of the theorem takes it into account.

It is convenient to make a slight change of the notation. Namely, instead of c_α we introduce $\lambda_\alpha : c_\alpha = \lambda_\alpha^2$ and rescale the parameters, $q^{AB} := q^{AB} \lambda_\alpha^{\mathrm{sign}(A-B)}$. Then the commutational relations will become more symmetric.

Fix now a number $\lambda \neq 0, \pm i$. Define a quantum category $\mathsf{Lin}_q^+(\lambda)$, with objects which are triples like $(V, Q, \pm 1)$ with the following relations

$$\begin{aligned} x^A x^B - q^{AB} \lambda^{\pm\,\mathrm{sign}(A-B)} x^B x^A = 0, \\ \xi^A \xi^B + (-1)^{\tilde{A}+\tilde{B}} q^{AB} \lambda^{\mp\,\mathrm{sign}(A-B)} \xi^B \xi^A = 0, \end{aligned} \tag{3.12}$$

(the basis in V is defined uniquely up to the scaling of the basis vectors). Then the matrix entries of the "homomorphisms from (V, Q_V, ε) to (W, Q_W, η)" are subject to the following commutational relations implied by (2.15-2.17) (3.12). For the elements of one row, if $K < L$:

$$\begin{aligned} t_A{}^K t_A{}^L - q^{KL}\lambda^{-\eta} t_A{}^L t_A{}^K &= 0, \quad \tilde{A} = 0, \\ t_A{}^K t_A{}^L + (-1)^{\tilde{K}+\tilde{L}} q^{KL}\lambda^{\eta} t_A{}^L t_A{}^K &= 0, \quad \tilde{A} = 1. \end{aligned} \tag{3.13}$$

For the elements of one column, if $A < B$:

$$\begin{aligned} t_A{}^K t_B{}^K - (q_{AB})^{-1}\lambda^{-\varepsilon} t_B{}^K t_A{}^K &= 0, \quad \tilde{K} = 0, \\ t_A{}^K t_B{}^K + (-1)^{\tilde{A}+\tilde{B}} (q_{AB})^{-1}\lambda^{\varepsilon} t_B{}^K t_A{}^K &= 0, \quad \tilde{K} = 1. \end{aligned} \tag{3.14}$$

For the "diagonal" elements $(A < B, K < L)$:

$$t_A{}^K t_B{}^L - (q_{AB})^{-1} q^{KL} (-1)^{\tilde{A}\tilde{L}+\tilde{B}\tilde{K}} t_B{}^L t_A{}^K = (q_{AB})^{-1} \frac{\lambda^{-\varepsilon-\eta} - \lambda^{\varepsilon+\eta}}{\lambda^{-\eta} + \lambda^{\eta}} (-1)^{(\tilde{A}+\tilde{B})\tilde{K}} t_B{}^K t_A{}^L \tag{3.15}$$

For the "antidiagonal" elements $(A < B, K > L)$:

$$t_A{}^K t_B{}^L - (q_{AB})^{-1} q^{KL} (-1)^{\tilde{A}\tilde{L}+\tilde{B}\tilde{K}} t_B{}^L t_A{}^K = (q_{AB})^{-1} \frac{\lambda^{-\varepsilon+\eta} - \lambda^{\varepsilon-\eta}}{\lambda^{-\eta} + \lambda^{\eta}} (-1)^{(\tilde{A}+\tilde{B})\tilde{K}} t_B{}^K t_A{}^L \tag{3.16}$$

Notice that for $\varepsilon = \eta$ the right-hand side of (3.16) vanishes, and for $\varepsilon = -\eta$ so do the right-hand side of (3.15). The factor in the right-hand side of (3.15) or (3.16) respectively reduces to $\lambda^{-\varepsilon} - \lambda^{\varepsilon}$ (in both cases).

Now if one will fix the classical space V, $\dim V = 2$ and change only the quantization parameters q^{AB} he will arrive to the example described in the introduction.

Appendix.

The general definition of a *bialgebra* $A = (A_i)$ *indexed by the set* I is as follows. We should fix the relations $P \subset I^3, Q \subset I^3$ and define the linear maps

$$\begin{aligned} m_{ijk} : A_i \otimes A_j &\longrightarrow A_k \text{ for all } (i,j,k) \in P, \\ \Delta_{xyz} : A_x &\longrightarrow A_y \otimes A_z \text{ for all } (x,y,z) \in Q, \end{aligned}$$

such that the diagram

$$\begin{tikzcd} A_i \otimes A_j \arrow[r, "m_{ijk}"] \arrow[d, "\Delta_{ixy}\otimes\Delta_{j\alpha\beta}"'] & A_k \arrow[r, "\Delta_{krs}"] & A_r \otimes A_s \\ A_x \otimes A_y \otimes A_\alpha \otimes A_\beta \arrow[rr, leftrightarrow] & & A_x \otimes A_\alpha \otimes A_y \otimes A_\beta \arrow[u, "m_{x\alpha r}\otimes m_{y\beta s}"'] \end{tikzcd}$$

is commutative when the maps in it make sense.

It is obvious that this definition is self-dual: if $(A_i, I, P, Q, m, \Delta)$ is a bialgebra, then $(A'_i, I, Q, P, \Delta', m')$ is also a bialgebra. This definition contains in particular: graded and almost graded algebras, coalgebras and bialgebras (in the usual sense), function algebras on categories, additive categories, function algebras on quantum categories, algebras dual to them, etc. Rather exotic diagonals one can find, for example, in the Odessky – Feigin algebras [25].

The possibility that need to be also considered is that the indices $i \in I$ may form not just a set but topological space or a smooth manifold. The natural farther step is to allow the indices to be coordinates on a supermanifold or even a "quantum manifold". In the context of this paper that means to allow to quantize the quantization parameters themselves.

References

[1] H.J.Baues. Algebraic homotopy. Cambridge University Press, Cambridge (1989).

[2] J.N.Bernstein, D.A.Leites. How to integrate the differential forms on supermanifolds. Funktsion. anal. pril.,**11**, no.1 (1977), 70-71.

[3] V.M.Buchstaber. The operator doubles and the semigroups of maps into groups. Doklady Rossijskoj Akademii Nauk, **341**, 6 (1995), 1-3.

[4] V.M.Buchstaber, E.G.Rees. Multivalued groups, their representations and Hopf algebras. Transactions of AMS. (1995) (To appear.)

[5] V.M.Buchstaber. Semigroups of maps into groups, operator doubles and complex cobordisms. (To appear.)

[6] Th.Voronov. Quantization on supermanifolds and the analytic proof of the Atiyah – Singer index theorem. In: Sovrem. probl. matematiki. Novejshije dostizh., Vol.38, VINITI, Moscow (1990), 3-119.

[7] Th.Voronov. On characteristic classes of infinite-dimensional vector bundles. Uspekhi matem. nauk, **46**,no.3 (1991), 185-186.

[8] Th.Voronov. Geometric Integration Theory on Supermanifolds. Sov.Sci.Rev.C. Math.Phys.,V.9 (1992), I-IV + 1-138.

[9] S.I.Gelfand, Yu.I.Manin. Methods of the homological algebra. Introduction to the cohomology theory and to derived categories. Nauka, Moscow (1988).

[10] G.W.Gibbons, S.W.Hawking. Selection rules for topology change. Commun. Math. Phys.,**148**, 2 (1992), 345-352.

[11] E.E.Demidov. Function algebras on quantum matrix supergroups. Funktsion. anal. pril., **24**, no.3 (1990), 78-79 .

[12] E.E.Demidov. On some aspects of the theory of quantum groups. Uspekhi matem. nauk, **48**, no.6 (1993), 39-74 .

[13] E.E.Demidov. Quantum groups. In: Algebra-10, VINITY, Moscow (in print).

[14] J.Donin, S.Shnider. Deformations of quadratic algebras and the corresponding quantum semigroups. Preprint, Bar-Ilan University (May 1995).

[15] M.V.Karasev, V.P.Maslov. Nonlinear Poisson brackets. Geometry and quantization. Nauka, Moscow (1991).

[16] D.G.Quillen. Higher algebraic K-theory.I. In: Lecture Notes Math.,341 (1973), 85-147.

[17] V.N.Latyshev. The combinatorial theory of rings. Standard bases. Moscow State University Press, Moscow (1988).

[18] Yu.Manin. Quantum groups and non-commutative geometry. Preprint / CRM-1561 (1988), Montreal.

[19] Yu.Manin.Multiparametric quantum deformation of the general linear supergroup. Commun. Math. Phys., **123**(1989), 163-175.

[20] Ch.Misner, K.Thorne, J.Wheeler. Gravitation. W.H.Freeman and Co. (1973).

[21] B.Mitchell. Rings with several objects. Adv. Math.,**8** (1972), 1-161.

[22] Yu.A.Neretin. The spinor representation of infinite-dimensional orthogonal semigroup and the Virasoro algebra. Funktsion. anal. pril., **23**, no.3 (1989), 32-44.

[23] Yu.A.Neretin. The extension of representations of the classic groups to the representations of categories. Algebra i analiz, **3**, no.1 (1991), 176-202.

[24] S.P.Novikov. Various doubles of the Hopf algebras. Operator algebras on quantum groups, complex cobordisms. Uspekhi matem. nauk, **47**, no.5 (1992), 189-190.

[25] A.V.Odessky, B.L.Feigin. Elliptic Sklyanin algebras. Funktsion. anal. pril., **23**, no.3 (1989), 45-54.

[26] N.Yu.Reshetikhin, L.A.Takhtajan, L.D.Faddeev. Quantization of Lie groups and Lie algebras. Algebra i analiz, **1**, no.1 (1989), 178-206.

[27] S.Shnider, S.Sternberg. Quantum groups. From coalgebras to Drinfeld algebras. International Press Inc. (1993).

[28] A.Sudbery. Consistent multiparameter quantization of $\mathrm{GL}\,(n)$. Preprint (1990).

[29] V.G.Turaev. The category of oriented tangles and its representations. Funktsion. anal. pril., **23**, no.3 (1989), 93-94.

[30] Ch.Watts. A homology theory for small categories. Proc. of the Conf. on Categorical Algebra. La Jolla, CA (1965).

[31] S.Eilenberg, S.MacLane. General theory of natural equivalences. Trans. of AMS, **58** (1945), 231-294.